A note on bobs-only Grandsire Triples

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Abstract. In 1886, Thompson presented proofs that (i) there is no bobs-only extent of Grandsire Triples, that (ii) there is no bobs-only touch of Grandsire Triples longer than 4998 changes, and that (iii), in the case of a bobs-only touch of 4998 changes, the remainder is a bob course. We argue that, although the results are supported by experience, his proofs of (ii) and (iii) are incorrect; however, a correct proof (without explicit criticism of Thompson's proof) of (ii) is to be found in 2016 work by van der Sluijs. We outline a minor variant of this proof and extend it with a proof, superseding Thompson's incorrect proof, of (iii). (There is no challenge to the correctness of his proof of (i).)

1 Introduction

Wilson [16] is a good (but imperfect) reference for many of the campanological terms not defined here. We consider mainly the method of Grandsire Triples, but we comment on other methods so that what is special about Grandsire Triples stands out. Our object is to show that an argument of Thompson [13] from 1886 about long bobs-only[1] touches of Grandsire Triples is incorrect.

We recall that (for Grandsire Triples) a *plain course* (if starting from rounds) consists[2] of the five rows[3]

$$1253746, 1275634, 1267453, 1246375, 1234567$$

and a *bob course*, also if starting from rounds, consists of the three rows

$$1752634, 1467352, 1234567.$$

There is however no requirement to start from rounds. Since Holt's work in 1751, it is known that there is a bobs-only touch of length 4998 (i.e. 357 rows of the kind considered here) and that the remainder is a bob course.

Thompson stated (and in the **first** case correctly proved) three results:

1. There is no bobs-only extent of Grandsire Triples;
2. Any bobs-only touch of Grandsire Triples has length at most 4998 changes;
3. The changes not included in a bobs-only touch of Grandsire Triples of length 4998 form a bob course.

As background we briefly outline a simple correct proof, following Dickinson [3], of the first of these three results. We then analyse where Thompson's arguments about **touches** break down. Van der Sluijs [12] correctly[4] proves the second of these three results; we present a minor variation of his argument and show how to extend it to prove the third result.

2 Standard terminology

We will use the symbol $\oplus_n$ to indicate addition (mod n): thus, $1 \oplus_5 2 = 3$, $2 \oplus_5 3 = 0$, etc.

Warning: there are two conventions about whether an operator is written on the left or the right of that which it operates on (the "operand"). Use of one convention rather than another is a matter of personal choice; we adopt that which seems most natural, namely the operator is written to the **right** of the operand. Some others, such as Rankin [9], do it differently. The results are essentially the same.

[1] We'd prefer to say "single-free"; but the terminology is already fixed.
[2] Augmented, of course, in each lead, by the other 13 changes of the lead.
[3] Backstroke changes with the treble leading.
[4] We have some minor reservations about his choice of terminology and notation, but this doesn't affect the argument.

2.1 Bells, Positions and Rows

The set $\{1, 2, \ldots, 7\}$ can be considered both as a set $\mathcal{B}$ of *bells* and as a set $\mathcal{P}$ of *positions*. It is conventional to use the same symbols in each case, although letters such as $A, B, \ldots, G$ might be acceptable for bells, and would make the conceptual distinction between bells and positions clear. A *row* is then a 1–1 map[5] from $\mathcal{B}$ to $\mathcal{P}$. Thus it puts each bell into a position, in such a way that every two distinct bells occupy distinct positions. As examples of rows, we have *rounds*, the row 1234567, and *back-rounds*, the row 7654321, which indicates the map that puts bell 7 into position 1, bell 6 into position 2, etc.

2.2 Operators, Cycle Notation and Examples

An *operator* is a permutation of the set $\mathcal{P}$ of positions, i.e. a 1-1 map from $\mathcal{P}$ to $\mathcal{P}$. If p is a position and O an operator, we write pO for the position to which O maps p.

There is a standard notation for permutations, and thus for such operators: it is[6] *cycle notation*, according to which, for example, the operator $p \mapsto ((p-1) \oplus_7 1) + 1$ is written as (1234567); it takes position 1 to position 2, position 2 to position 3, etc, and position 7 to position 1. The simplest operator is the *identity* operator I, with $pI = p$ for each position p. In *full disjoint cycle notation* it is $(1)(2)(3)(4)(5)(6)(7)$—"full" means that each position is mentioned at least once, "disjoint" means that each position is mentioned at most once. A *cycle* is a permutation whose cycle notation can be written using just one parenthesised group.

As another example we have (in Grandsire Triples) the *Plain* operator $P = (34675)$, which takes position 3 to position 4, position 4 to position 6, etc, and, finally, position 5 to position 3. It is a cycle; it may also be written more fully as $(1)(2)(34675)$.

As yet another example we have (in Grandsire Triples) the *Bob* operator $B = (247)(365)$, which takes position 2 to position 4, position 4 to position 7, position 7 to position 2, etc. It is not a cycle; it may also be written more fully as $(1)(247)(365)$, which has the virtue of being easily memorable.

2.3 Composition of Operators, Associativity and Inverses

Operators can be *composed*: if $O : \mathcal{P} \to \mathcal{P}$ and $O' : \mathcal{P} \to \mathcal{P}$ are operators, then their *composite* is the operator $OO' : \mathcal{P} \to \mathcal{P}$ defined by $p(OO') = (pO)O'$. Composition is *associative*, i.e. $(OO')O'' = O(O'O'')$ for all operators O, O' and O'': so we don't need parentheses in writing the composite $OO'O''$. If $O_1, \ldots, O_n$ is a sequence of operators then its *composite* is the operator $O_1O_2 \ldots O_n$ obtained by composing the operators in the sequence. By convention, $O^0 = I$ and, for $n \geq 0$, $O^{n+1} = OO^n$. It is easy to show that $IO = OI = O$ for every operator O. It is rarely true that $OO' = O'O$. Every operator O has an *inverse* O^{-1}, so that $OO^{-1} = I = O^{-1}O$.

Since the number of positions is finite, the number of operators is finite—in this case it is 7!; some easy mathematics in fact shows that for every operator O there is a smallest number $n > 0$, the *order* of O, such that $O^n = I$, and that (for Grandsire Triples) this number is at most 12, as exemplified by the operator $(123)(4567)$. The Plain operator P has order 5; the Bob operator B has order 3. Note that $O^{n-1} = O^{-1}$ when n is (or is a multiple of) the order of O. A cycle of order n is called an *n-cycle*; thus, P is a 5-cycle.

By composing operators in a different fashion one can have an equivalent theory, except that convenient equations like $p(OO') = (pO)O'$ become, for example, $p(OO') = (pO')O$, which is much less convenient.

2.4 Action of Operators on Rows

Operators *act* on rows: if r is a row and O is an operator, then rO is the row obtained by the composition of the two maps $r : \mathcal{B} \to \mathcal{P}$ and $O : \mathcal{P} \to \mathcal{P}$. It easily follows that if r is a row then $r(OO') = (rO)O'$.

For example, if r is the row 1234567, then rP is the row 1253746 and rB is the row 1752634. Likewise, if r is the row 1532467, then rP is the row 1543726 and rB is the row 1745632. When r is written out as a row, we write (e.g.) $1532467 \cdot P$, with a $\cdot$ for typographical clarity.

2.5 Touches

A *bobs-only touch* is a non-empty sequence of the operators P and B, whose composite is the identity operator I, such that no proper subsequence is a touch[7]. Two simple examples are P, P, P, P, P (often regarded as too

[5] [12] considers rows to be 1–1 maps from $\mathcal{P}$ to $\mathcal{B}$.

[6] We ignore the alternative notation according to which an operator is given by its action on rounds.

[7] In other words, no change occurs twice.

simple to be called a "touch") and B, B, B. We also use the word *touch* for the sequence of rows obtained by applying the operators to rounds. The phrase "bobs-only" indicates that we are not using the *Single* operator $S = (247365)$.

A more complex example is the touch $B, B, P, B, B, P, B, B, P, B, B, P, B, B, P$ (leading to 210 changes); that "no change occurs twice" amounts to the 120 assertions that, starting from the m^{th} operator and continuing to the $(m+n)^{th}$ operator, the composite is not equal to the identity I. (Of course, the symmetry and repetition in this touch allows the checking of 120 assertions to be reduced substantially.)

From now on we restrict attention to rows and operators that are even[8] permutations of $\{1, \ldots, 7\}$ and that fix 1 (and so we could omit the occurrence of 1, so long as it is understood that the positions left are numbered upwards from 2). There are 360, i.e. $6!/2$, such rows and the same number of operators.

Thompson [13] correctly shows that there is no touch of length 360; we outline below a simplification of his argument, due to Dickinson [3]. In mathematical terms, one may restate his result as "There is no sequence of P and B that unicursally traverses the Cayley graph of the finite permutation group with P and B as generators".

He also argues that there is no bobs-only touch of length greater than 357; that there is such a touch of length 357 is due to Holt in 1751. His result is correct [12]; we will present a local counterexample to his argument and then a proof of the result's correctness.

2.6 Two Important Conditions

The methodology that we inherit from [13] is to consider only the treble's backstroke leads; it is a nice feature of Grandsire Triples that (whether there is or is not a bob) each such stroke is related to its immediate precursor at handstroke by a **fixed** odd permutation, namely, in cycle notation, (1)(23)(45)(67). Thus, if we find a way of generating (without falsehood) all the even changes with treble leading at the backstroke, then all the immediately preceding changes (those at handstroke) will constitute the desired falsehood-free sequence of odd changes.

In a different method, such as Plain Bob, the difference between the effects of P and B on the treble's handstroke means that this argument doesn't apply; but a different one (not applicable to Grandsire) does apply: consider the change from a treble's backstroke to the treble's next handstroke. In Plain Bob Doubles this is the even permutation (1)(23)(45), in Plain Bob Minor it is again the even permutation (1)(23)(45)(6), in Plain Bob Triples it is the odd permutation (1)(23)(45)(67) and in Plain Bob Major it is the odd permutation (1)(23)(45)(67)(8).

Thus, the arguments below apply only if the change from a treble's backstroke lead to the previous change is (as in Grandsire Triples) a fixed odd permutation, or the same is true (as in Plain Bob Triples or Major) of the change to the next treble handstroke lead.

3 Some Elementary Results

The following elementary result is essential (but first recall that operators are just permutations):

Lemma 3.1. *Let θ be a permutation of $\{1, 2, \ldots, n\}$, expressed in full disjoint cycle notation using m cycles. Then θ is an even permutation iff the number*[9] *$n - m$ is even.*

Proof. The cycles of even order make an even contribution to n; so the number of cycles of odd order is odd if and only if n is odd. These odd order cycles however are even permutations. Those of even order are odd permutations; their number determines the parity of θ. □

An example may help: take $n = 5$, and consider the two permutations (1)(23)(45) and (1234)(5), respectively even and odd. The first has $m = 3$ and so $n - m = 2$, an even number; the second has $m = 2$, and so $n - m = 3$, an odd number.

Using full disjoint cycle notation, this gives a simple way to determine the parity of a permutation. Note that, as we are just interested in the parity, we can use $n + m$ rather than $n - m$, since $2m$ is even. If we have a permutation acting instead on $\{0, 1 \ldots, n - 1\}$, it is routine to make the appropriate changes to the result of the Lemma.

[8] [13] says "in course".
[9] The *discriminant* of θ.

Likewise, one may show [5, Thm 5.1.2] that the order of a permutation is the least common multiple of the lengths of the cycles in a disjoint cycle representation. For example, the even permutation (1)(23)(45) has order 2; the odd permutation (1234)(5) has order 4; and the odd permutation (1234)(567) has order 12. Permutations of even order can be even or odd, as in these examples; those of odd order are always even.

4 Simpler Problems

Before diving into the details of Grandsire Triples, we consider some related problems:

4.1 Plain Bob Doubles

In this method instance, $P = (2453)$ and $B = (45)$, both odd. The change at the lead-end to the lead-head varies in accordance with whether or not there is a bob. The change from any lead-head to the next lead-end (just before the next lead-head) is the even permutation $R = (23)(45)$. So, if we look for a bobs-only touch by considering only the lead-heads, we are in danger of falsity, both because P and B are odd permutations and because R is even. The analysis below for Grandsire Triples is therefore inapplicable. The touch $PPPBPPPBPPPB$ (i.e. $PPPB$ repeated twice) is a true extent, with lead-heads alternately odd and even.

4.2 Plain Bob Minor

In this method instance, $P = (24653)$ and $B = (465)$, both even. The change from any lead-head to the next lead-end (just before the next lead-head) is the even permutation $R = (23)(45)$. The evenness of P and B means that as we build up operators such as PP, PPB, $PPBPPB$ etc from P and B they will all be even, as also is R, so we cannot hope to reach any of the odd rows (with the treble leading) in an extent, but maybe a lesser objective can be achieved—a "bobs-only' 360, i.e. 30 leads.

It can, but, because of the evenness of R, its construction needs to avoid repetition of any row either at a lead-head or at a lead-end. One standard bobs-only touch of 360 is achieved by calling the 6 "Wrong", "Home" and "Wrong", repeated twice. In our notation it is $BPPPBBPPPP$, repeated twice.

4.3 Plain Bob Triples

In this method instance, $P = (246753)$ and $B = (4675)$, both odd. The change from any lead-head to the next lead-end (just before the next lead-head) is the odd permutation $R = (23)(45)(67)$. The oddness of P and B means that, as we build up operators such as PP, PPB, $PPBPPB$ etc from P and B, half will be even and half odd, exactly depending on the parity of the number of operators. The analysis below for Grandsire Triples is therefore inapplicable; in fact, a bobs-only extent of 5040 is achievable, e.g. the 5-part composition by Henry Hubbard at [7].

4.4 Plain Bob Major

In this method instance, $P = (2468753)$ and $B = (46875)$, both even. The change from any lead-head to the next lead-end (just before the next lead-head) is the odd permutation $R = (23)(45)(67)$. So, if a sequence of P and B operators of length 2,520 (i.e. $40,320/16$) generates, one-by-one, all the 2,520 even rows with the treble leading (as lead-heads), then (15 changes later) their variants by R will consist of all the odd rows with the treble leading. The question may therefore be posed: can it be achieved? (Practical questions such as "Is it musical?", "Is it easy to conduct?", "Is there a band with the stamina to ring it?" are ignored in this paper.) We return to this after considering Grandsire Triples.

5 Round Blocks and Decompositions

A *round block* [13] is a repetition-free sequence of rows (so each begins with 1), each obtained from its predecessor[10] by one use either of P or of B. The first row is regarded as immediately following the last row, i.e. the first row is regarded as the *successor* of the last row[11]. In such a block, we say that a row x is *Plained* if

[10] If r is the predecessor of r', then r' is the successor of r, and vice-versa.

[11] Without this condition, the block could hardly be said to be "round". However, all blocks we consider will be round, and for brevity we omit the word.

its successor is xP, and *Bobbed* if its successor is xB. The *order* of the block is just the number of elements it contains. A (bobs-only) touch is just a round block that ends in rounds.

For example, the plain course displayed above, obtained just by using P, is a round block of order 5. Likewise, the displayed bob course is a round block of order 3, obtained just by using B. Longer examples may be obtained using any bobs-only touch and an arbitrary row as starting point. For example, the sequence $P, P, B, P, P, B, P, P, B, P, P, B$ gives such a touch[12] , and hence a round block of order 12. (It may be observed that many, if not most, touches of this Triples method use singles.)

A *decomposition* [3] is a disjoint[13] collection $\mathcal{D}$ of round blocks whose orders add up to 360; it decomposes the set of 360 rows into round blocks. For example, one can have 72 blocks each made up with use only of P, or 120 blocks each made up with use only of B. A disjoint collection of round blocks with total order less than 360 may or may not be extendable to a decomposition: see Section 7 below.

A bobs-only extent (if such a thing existed) would correspond, since $5040 = 360 \times 14$, to a decomposition consisting of a single round block of order 360, ending with *rounds*, the row 1234567. But, no such thing exists; we shall present the argument.

6 Q-sets and Q-cycles

We follow [13] by defining the operator Q as the composite of P and B^{-1}. But, since we are writing operators on the right rather than on the left it is PB^{-1} rather than Thompson's $B^{-1}P$. Note that $QB = (PB^{-1})B = P(B^{-1}B) = PI = P$. Since $P = (1)(2)(34675)$ and $B^{-1} = (1)(274)(356)$, we can calculate that $Q = (1)(27643)(5)$. Q has order 5, i.e. we have, (for any row x), $xQ^5 = x$ (and this holds for no smaller positive index).

Definition 6.1. *Two rows x and y are* Q-equivalent*, written $x \equiv_Q y$, iff for some i we have $x = yQ^i$; the relation of being Q-equivalent is an equivalence*[14] *relation, and so it partitions the 360 rows into $72 = 360/5$ equivalence classes, called* Q-sets.

Thus, the Q*-set* of a row x is the set $\{xQ^i : 0 \le i \le 4\}$, of size 5. For example, the Q-set containing the row 1234567 is the set

$$\{1234567, 1346572, 1467523, 1672534, 1723546\}.$$

Definition 6.2. *A* Q-cycle *is a cyclic sequence $(xQ^i : 0 \le i \le 4)$, treated as an ordered set with each xQ^i immediately followed by xQ^{i+1} (and with the convention that x immediately follows xQ^4).*

It is useful to have a *canonical representative* from each Q-set. Note that all elements of the Q-set have the same bell in position 5, so we can begin by choosing that bell from the bells $\{2, \ldots, 7\}$. Let us call it V (for "piVot"—or recall the Roman symbol for 5). Then, we consider the predecessor V' of V, e.g. if V is 3 we consider 2, with the convention that 7 is the predecessor of 2. [Other choices are possible here.] Among the five members of the Q-set there is exactly one that has this bell V' in position 4. This is the representative we choose. It can be abbreviated by the information consisting of V and, in order, the bells in positions 2 and 3. Rounds is then, as one would expect and hope, the representative of the Q-set to which it belongs.

Here is a systematic enumeration of twelve (all those with 7 and 2 in positions 4 and 5) of the canonical representatives of the 72 Q-sets:

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[1, 3, 4, 7, 2, 6, 5]  [1, 4, 3, 7, 2, 5, 6]  [1, 5, 3, 7, 2, 6, 4]  [1, 6, 3, 7, 2, 4, 5]
[1, 3, 5, 7, 2, 4, 6]  [1, 4, 5, 7, 2, 6, 3]  [1, 5, 4, 7, 2, 3, 6]  [1, 6, 4, 7, 2, 5, 3]
[1, 3, 6, 7, 2, 5, 4]  [1, 4, 6, 7, 2, 3, 5]  [1, 5, 6, 7, 2, 4, 3]  [1, 6, 5, 7, 2, 3, 4]
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That done, we can represent any row by a pair consisting of the canonical representative of its Q-set and the number of applications of Q required to reach it from that representative. For example, 1765234 is thus represented by the pair $(1347265, 2)$. To work this out, note that the row 1765**2**34 has bell 2 in position 5; so one just applies I, Q, Q^2, etc to the row until one gets the 7 (agreed to be the predecessor of 2) into position 4. In this case we have $1765234 \cdot Q^3 = 134\mathbf{72}65$, whence $1347265 \cdot Q^2 = 1765234$; we thus obtain the two required components of the pair.

[12] It is 1253746, 1275634, **1462375**, 1436527, 1453762, **1274653**, 1267345, 1236574, **1452736**, 1475623, 1467352, **1234567**, where rows just reached by a Bob are in bold font. Their immediate predecessors are those that are Bobbed.

[13] i.e. there are no overlaps: no row appears in more than one round block.

[14] i.e. it is reflexive, symmetric and transitive.

7 Decompositions and Q-sets

The following result emphasises the importance of decompositions:

Theorem 7.1. *There is a natural 1–1 correspondence between decompositions and the maps from the set $\mathcal{Q}$ of Q-sets to the set $\{P, B\}$.*

Proof. Let $\mathcal{D}$ be a decomposition. Wherever it contains some round block R which contains a row x immediately followed by xP, the row x is plained in R, and thus is *plained* in D; since D contains all the 360 rows[15], it must also contain xQ, and if this is bobbed (either in R or in some other round block R'), then $(xQ)B$, i.e. $x(QB)$, i.e. xP, will immediately follow it—so xQ must be plained in D, lest the decomposition $\mathcal{D}$ contain a repetition. So, for each x, the Q-set of x is either all plained in D or all bobbed in D, i.e. we have a map $f_{\mathcal{D}}$ from $\mathcal{Q}$ to $\{P, B\}$.

Conversely, suppose we have such a map $f : \mathcal{Q} \to \{P, B\}$; we construct a decomposition $\mathcal{D}_f$ as follows. Consider any row x; it is in exactly one Q-set Q_x, and then $f(Q_x)$ equals either P or B, which determines whether x should be plained or bobbed in $\mathcal{D}_f$. If the plaining (by this method) of x to xP and the bobbing of y to yB lead to $xP = yB$, then $y = xP(B^{-1}) = x(PB^{-1}) = xQ$, so y and x are in the same Q-set, and so must be both plained or both bobbed by f, contrary to hypothesis; hence, as required for construction of a decomposition, there are no repetitions. This approach covers all rows, so we obtain a decomposition. □

Thus, so long as we are just interested in decompositions, once the nature (plained or bobbed) of each of the 72 Q-sets (or just of their canonical representatives) is determined, the nature (plained or bobbed) of each row is determined. For example, if all Q-sets are plained, we obtain a decomposition consisting of $72 = 360/5 = 5040/70$ disjoint round blocks, each of order 5; if all are bobbed, then we obtain a decomposition consisting of $120 = 360/3 = 5040/42$ disjoint round blocks, each of order 3. Note that 72 is an even number; it is coincidental (but not useful) that it is also the number of Q-sets.

8 Chains

Let $\mathcal{D}$ be a decomposition. Recall that $\mathcal{D}$ determines whether all elements of any Q-set are plained or all are bobbed. Consider a Q-set Q_0.

Definition 8.1. *A* chain *(for the pair $(\mathcal{D}, Q_0)$) is a subsequence of one of the round blocks in $\mathcal{D}$ such that*

(i) its first element is the successor (according to $\mathcal{D}$) of an element of Q_0;

(ii) its last element is an element of Q_0; and

(iii) no other element of the chain is in Q_0.

Choose a member r_0 of Q_0; it could, but need not, be the canonical representative studied above. For each $i = 0, 1, 2, 3, 4$, define[16] C_i to be the unique chain of rows determined by $\mathcal{D}$, up to and including r_0Q^i and starting from **just after** whichever $r_0Q^{\gamma(i)}$ is in the same block. That it is a chain means, *inter alia*, that the only element of Q_0 that it contains is r_0Q^i. Since each chain ends with one of the r_0Q^i and contains no other r_0Q^j, we have exactly five such chains: C_0, C_1, C_2, C_3 and C_4.

Definition 8.2. *This determines the* associated permutation γ *of the set $\{0, 1, 2, 3, 4\}$. Note that $r_0Q^{\gamma(i)}$ may be r_0Q^i itself.*

The reader who has understood the above definitions should skip to the Proposition at the end of this section; for others, we give some examples to illustrate (and, hopefully, clarify) matters:

Example 8.1. *Let $\mathcal{D}$ be the decomposition with all the Q-sets plained. That gives us 72 round blocks, each of size 5. Consider the Q-set*

$$Q_0 = \{r_0, r_0Q, r_0Q^2, r_0Q^3, r_0Q^4\} = \{1234567, 1346572, 1467523, 1672534, 1723546\},$$

where r_0 is rounds.

[15] i.e. the even lead heads.

[16] The literature has an inequivalent but similar definition; our choice ensures that the numbering of chains is independent of whether Q_0 is plained or bobbed in $\mathcal{D}$.

We calculate the chains for the pair $(\mathcal{D}, Q_0)$. We obtain the five chains (with the elements of Q_0 in bold):

$$C_0 = [r_0P,\ r_0P^2,\ r_0P^3,\ r_0P^4,\ \mathbf{r_0}] = [1253746, 1275634, 1267453, 1246375, \mathbf{1234567}],$$

$$C_1 = [r_0QP,\ r_0QP^2,\ r_0QP^3,\ r_0QP^4,\ \mathbf{r_0Q}] = [1354267, 1325746, 1372654, 1367425, \mathbf{1346572}],$$
$$C_2 = [r_0Q^2P,\ r_0Q^2P^2,\ r_0Q^2P^3,\ r_0Q^2P^4,\ \mathbf{r_0Q^2}] = [1456372, 1435267, 1423756, 1472635, \mathbf{1467523}],$$
$$C_3 = [r_0Q^3P,\ r_0Q^3P^2,\ r_0Q^3P^3,\ r_0Q^3P^4,\ \mathbf{r_0Q^3}] = [1657423, 1645372, 1634257, 1623745, \mathbf{1672534}],$$
$$C_4 = [r_0Q^4P,\ r_0Q^4P^2,\ r_0Q^4P^3,\ r_0Q^4P^4,\ \mathbf{r_0Q^4}] = [1752634, 1765423, 1746352, 1734265, \mathbf{1723546}].$$

Each of these chains happens to be a round block—we have chosen a very very simple case. The permutation γ is equally simple, with $\gamma(i) = i$ for all i, so $\gamma = (0)(1)(2)(3)(4)$. Note that this expression uses 5 cycles; and 5 round blocks are involved.

Example 8.2. *Let $\mathcal{D}'$ be the decomposition with all the Q-sets plained except for the set Q_0 considered in the previous example, which is now bobbed.*

$$Q_0 = \{r_0, r_0Q, r_0Q^2, r_0Q^3, r_0Q^4\} = \{1234567, 1346572, 1467523, 1672534, 1723546\},$$

where r_0 is rounds.

We calculate the chains for the pair $(\mathcal{D}', Q_0)$. Note that $Q^4P = B$, $Q^3P = Q^4B$, etc. We obtain the five chains (with the elements of Q_0 in bold):

$$C_0 = [r_0P,\ r_0P^2,\ r_0P^3,\ r_0P^4,\ \mathbf{r_0Q^0}] = [1253746, 1275634, 1267453, 1246375, \mathbf{1234567}],$$

$$C_4 = [r_0Q^4P,\ r_0Q^4P^2,\ r_0Q^4P^3,\ r_0Q^4P^4,\ \mathbf{r_0Q^4}] = [1752634, 1765423, 1746352, 1734265, \mathbf{1723546}],$$
$$C_3 = [r_0Q^3P,\ r_0Q^3P^2,\ r_0Q^3P^3,\ r_0Q^3P^4,\ \mathbf{r_0Q^3}] = [1657423, 1645372, 1634257, 1623745, \mathbf{1672534}],$$
$$C_2 = [r_0Q^2P,\ r_0Q^2P^2,\ r_0Q^2P^3,\ r_0Q^2P^4,\ \mathbf{r_0Q^2}] = [1456372, 1435267, 1423756, 1472635, \mathbf{1467523}],$$
$$C_1 = [r_0QP,\ r_0QP^2,\ r_0QP^3,\ r_0QP^4,\ \mathbf{r_0Q}] = [1354267, 1325746, 1372654, 1367425, \mathbf{1346572}].$$

These chains are the same as before, and they are numbered the same; but they now constitute (in the order shown) a single round block (which includes all elements of Q_0). The numbering is obtained from the condition that each chain C_i ends in r_0Q^i.

The permutation γ is quite simple, with $\gamma = (01234)$. This can be seen from the way in which chain C_1 follows C_2, so $\gamma(1) = 2$, etc. Expressed otherwise, the ordering of the chains in making up a round block is $C_0C_4C_3C_2C_1$. Note that this expression for γ uses 1 cycle; and 1 round block is involved.

$\mathcal{D}'$ has 67 round blocks of size 5 and one of size 25: note that $67 \times 5 + 1 \times 25 = 360$.

Example 8.3. *Let $\mathcal{D}'$ be again the decomposition with all but one of the Q-sets plained. Without loss of generality, the bobbed Q-set Q^* is again that containing rounds:*

$$\{1234567, 1346572, 1467523, 1672534, 1723546\}.$$

Exactly as in the previous example, $\mathcal{D}'$ has 68 round blocks, each of size 5 except for one of size 25, with $67 \times 5 + 1 \times 25 = 360$.

Let Q_0 now be the Q-set with canonical representative $r_1 = 1275634$*; its elements are* $r_1Q^0, r_1Q^1, r_1Q^2, r_1Q^3$ *and* r_1Q^4*, i.e.*

$$\{1275634, 1753642, 1534627, 1342675, 1427653\}.$$

We calculate the chains for the pair $(\mathcal{D}', Q_0)$. C_0 is exactly all the rows in the displayed round block of the previous example, starting just after and ending at $r_1Q^0 = 1275634$. *The other four are plain courses starting just after the remaining four members of the Q-set, namely* 1753642, 1534627, 1342675 *and* 1427653. *That is a total of five chains, each making a single round block. The permutation γ is just the identity, i.e.* $(0)(1)(2)(3)(4)$ *using five cycles. Note that five round blocks are involved.*

Example 8.4. *We continue the previous example by bobbing Q_0. This generates the decomposition $\mathcal{D}''$. The successors (obtained by a Bob) of the five elements of Q_0 will be*

$$\{1462375, 1267453, 1765234, 1563742, 1364527\}$$

and we obtain the following five chains for the pair $(\mathcal{D}'', Q_0)$:

$$\begin{aligned}
C'_4 &= [1462375, 1436527, 1453762, 1475236, \mathbf{1427653}],\\
C'_3 &= [1364527, 1356742, 1375264, 1327456, \mathbf{1342675}],\\
C'_2 &= [1563742, 1576234, 1527463, 1542376, \mathbf{1534627}],\\
C'_1 &= [1765234, 1726453, 1742365, 1734526, \mathbf{1753642}],\\
C'_0 &= [1267453, 1246375, \mathit{1234567}, 1752634, 1765423,\\
&\quad\ 1746352, 1734265, \mathit{1723546}, 1657423, 1645372,\\
&\quad\ 1634257, 1623745, \mathit{1672534}, 1456372, 1435267,\\
&\quad\ 1423756, 1472635, \mathit{1467523}, 1354267, 1325746,\\
&\quad\ 1372654, 1367425, \mathit{1346572}, 1253746, \mathbf{1275634}]
\end{aligned}$$

which constitute one round block of size 45. (Elements of Q^* are in italic.) 63 other round blocks are of size 5 and contain no elements of either of the two Q-sets bobbed so far. $63 \times 5 + 1 \times 45 = 360$.

The associated permutation γ can be calculated thus, using $r_1 = 1462375$ as the starting point: $r_1 Q^0 = 1275634$ and the previous element of Q_0 in the round block is 1753642, i.e. $r_1 Q^1$, and the next before that is 1534627, i.e. $r_1 Q^2$, etc; so γ is again the permutation (01234) *(in full disjoint cycle notation).*

Different results would be obtained, of course, if we were to calculate the chains for $(\mathcal{D}'', Q^*)$. C'_0 would be split; the others would be amalgamated.

This particular round block corresponds to the composition, starting from and ending with rounds, of

BPPPP, BPPPP, BPPPP, BPPPP, BPBPP, PPBPP, PPBPP, PPBPP, PPBPP

where the commas and spaces serve merely to separate the calls into groups of five.

We give a final example where the changes are suppressed, but indicating how one works it all out.

Example 8.5. *Suppose we have a decomposition, including two round blocks that, split into chains for some plained Q-set, are $C_0C_1C_3$ and C_2C_4.*

So, the first round block is

$$\begin{aligned}
C_0 &= [r_0Q^3P, \dots, r_0Q^0],\\
C_1 &= [r_0Q^0P, \dots, r_0Q^1],\\
C_3 &= [r_0Q^1P, \dots, r_0Q^3]
\end{aligned}$$

and the second is

$$\begin{aligned}
C_2 &= [r_0Q^4P, \dots, r_0Q^2],\\
C_4 &= [r_0Q^2P, \dots, r_0Q^4]
\end{aligned}$$

For each i, the chain C_i ends in r_0Q^i as required. The chain C_3 after C_1 begins at the last element of C_1, plained, i.e. r_0Q^1P; similarly for all the other chains.

Now we bob the Q-set. We have the same chains as before, with the same numbers, but rearranged. We recall the equations

$$Q^0B = Q^4P, \quad Q^2B = Q^1P, \quad Q^3B = Q^2P, \quad Q^4B = Q^3P, \quad Q^1B = Q^0P.$$

The chain after C_0 must now begin at $r_0Q^0B = r_0Q^4P$; so it is C_2 and therefore ends with r_0Q^2. Arguing similarly, using (in turn) the displayed equations, we get one round block

$$\begin{aligned}
C_0 &= [r_0Q^4B, \dots, r_0Q^0],\\
C_2 &= [r_0Q^0B, \dots, r_0Q^2],\\
C_3 &= [r_0Q^2B, \dots, r_0Q^3],\\
C_4 &= [r_0Q^3B, \dots, r_0Q^4]
\end{aligned}$$

and a second round block

$$C_1 = [r_0Q^1B, \dots, r_0Q^1].$$

The above is intended to illustrate that

1. chains depend both on a decomposition and on a Q-set of interest;
2. changing a Q-set from Plain to Bobbed
   (a) changes the decomposition;
   (b) can split round blocks apart;
   (c) can combine them together;
   (d) but leaves the chains intact (and[17] with the same numbering);
3. changing attention to a different Q-set will
   (a) break chains apart into shorter chains;
   (b) combine chains to make longer chains;

   and to suggest that
4. the permutation γ associated with a decomposition (and a Q-set) not only records how the chains are organised into round blocks (those that intersect the Q-set) but also has a representation in full disjoint cycle notation that tells us how many round blocks there are in the decomposition[18].

This last point is so important that we present it as follows:

Proposition 8.1. *[9, 3]. Let $\mathcal{D}$ be a decomposition and Q_0 a Q-set. Let γ be the associated permutation, determined by choosing to start the enumeration of the elements of Q_0 at its canonical representative[19]. Then (i) the round blocks that intersect Q_0 correspond to cycles in the full disjoint cycle representation of γ and, (ii) for each such block, its composition as a sequence of chains corresponds to the construction of the corresponding cycle as a cyclic sequence of elements of $\{0, 1, 2, 3, 4\}$.*

Proof. Immediate from the definitions. □

9 Thompson's Theorem

Theorem 9.1. *[13] In Grandsire Triples, let $\mathcal{D}$ be a decomposition. If the Q-sets $Q_1, \ldots, Q_n$ are bobbed in $\mathcal{D}$ and the Q-sets $Q_{n+1}, \ldots, Q_{72}$ are plained in $\mathcal{D}$, then the bobbing of Q_{n+1} alters the number of round blocks in $\mathcal{D}$ by an even number.*

Proof. [3] We look at the effect on the round blocks $R_1, \ldots, R_m$ that intersect Q_{n+1}; as illustrated above, the bobbing of Q_{n+1} breaks these into chains and puts the chains together again[20].

Let r be the canonical representative of Q_{n+1}; so $Q_{n+1} = \{rQ^0, rQ^1, rQ^2, rQ^3, rQ^4\}$, with $rQ^0 = r$. Each of these 5 rows belongs to a round block: for $0 \leq i < 5$, let $R_{j(i)}$ be the round block to which rQ^i belongs. This determines a useful map $j : \{0, \ldots, 4\} \to \{1, \ldots, m\}$.

As described above, $\mathcal{D}$ and Q_{n+1} determine five chains and a permutation[21] γ; for each i, the *predecessor* of chain C_i is $C_{\gamma(i)}$. Let α be the inverse γ^{-1} of γ: so for each i, $C_{\alpha(i)}$ is the *successor* of C_i. The chains are then just C_0, $C_{\alpha(0)}$, $C_{\alpha^2(0)}$, …, $C_{\alpha^4(0)}$. For each i, the last element of the chain C_i is rQ^i, and, since Q_{n+1} is Plained, rQ^iP is the first element of the successor $C_{\alpha(i)}$ of C_i.

Now bob the Q-set Q_{n+1}. This gives us a new decomposition $\mathcal{D}'$ which, with Q_{n+1}, also determines the same five chains but a new associated permutation γ'.

The first element of the new successor of C_i is $(rQ^i)B = r(Q^{i-1}(QB)) = r(Q^{i-1}P) = (rQ^{i-1})P$. But this is just the first element of the old successor $C_{\alpha(i-1)}$ of C_{i-1}.

[17] Thanks to our choice of notation, at variance with the literature.

[18] Dickinson [3] expresses this thus: "The number of cycles in the substitution … is obviously the number of round blocks of … that contain elements of the Q-set under consideration", in which "obviously" is the key word.

[19] Any other element will do; the point is not to change the choice.

[20] The basic argument is that of Thompson in [13], but, by the use of chains (from [9], and implicit in [3]), his case analysis is avoided. Rankin [9] is corrected in [10]; it has heavy use of notation and of group theory. McGuire [8] presents a simplified version of Rankin's argument.

[21] Alas, the notation $\gamma(i)$ here uses the rival convention that operators (such as γ) are applied on the **left** of their operands.

So whereas $C_{\alpha(i)}$ was the successor of C_i, now it is $C_{\alpha(i-1)}$ that is the successor. Thus we have split the (at most five) blocks $R_{j(i)} : i = 0, \ldots, 4$ into chains and rearranged the chains, making up blocks in a different way.

Define β by $\beta(i) = \alpha(i-1) \pmod 5$[22]. Then, after the bobbing, the chain $C_{\beta(i)}$ immediately follows[23] the chain C_i. Let π be the permutation (43210), i.e. the operation on the numbers $0, \ldots, 4$ that subtracts 1 (mod 5); then $\beta = \pi\alpha$ (first apply π, then apply α). Note that π is even, being a 5-cycle, so α and β are either both even or both odd.

By the Lemma, twice, the total number of cycles in α and the total number of cycles in β are both even or both odd. By the Proposition, these numbers correspond to the numbers of round blocks intersected by the Q-set, before and after the bobbing. Other round blocks are unaffected. Therefore, the bobbing of the Q-set does not change the evenness or oddness of the total number of round blocks. □

Corollary 9.1. *[13]. In Grandsire Triples, let $\mathcal{D}$ be a decomposition. Then its size is an even number.* □

Corollary 9.2. *[13]. There is no bobs-only extent of 5040 Grandsire Triples.* □

Corollary 9.3. *[14]. There is no bobs-only extent of 5040 Union Triples.*

Proof. As observed by Rankin [9], the same argument applies, with the variations that $P = (36475)$, $B = (247)(365)$ and (consequently) $Q = PB^{-1} = (276)$. □

As observed by Rankin [9], the same argument also applies to

1. Double Norwich Court Bob Major, where $P = (2745638)$, $B = (2637458)$ and so $Q = (235)$;
2. Grandsire Cinques, where $P = (34680E975)$, $B = (248E7)(36095)$ and so $Q = (279E08643)$ of order 9;

and one may add

1. Plain Bob Major, where $P = (2468753)$, $B = (46875)$ and so $Q = (253)$;
2. Union Cinques (rarely if ever rung, so we omit the details);
3. Double Norwich Court Bob Maximus, where $P = (2074T56E389)$, $B = (2T56074E389)$ and so $Q = (264)$;
4. Plain Bob Maximus, where $P = (24680TE9753)$, $B = (4680TE975)$ and so $Q = (253)$.

We are considering only method instances, (as in Grandsire Triples and Cinques but not Caters), where the lead end and the immediately following change, the lead head, are of opposite parity—so only the lead heads in the extent need to be considered. The demand for extents in most of these is low.

Mutatis mutandis, the same argument shows that if Q is of even order then the conclusion fails (or, rather, suggests that maybe there is a bobs-only extent but doesn't guarantee it or show one how to find it).

Wilson [16, p. 127] mentions Thompson's result (that there is no bobs-only extent), without proof, but offers a "simple way" to show that it is "unlikely" to be false: the common addition of extra courses to a plain course, by calling "a set of bobs on the same three bells", adds (he says) two courses to the plain course, and yet 72 (=5040/(5*14)) is even[24]. We do not regard this (even if a different touch is used) as remotely convincing. Still less convincing is his claim [16, p. 128] that the same reasoning applies to Stedman: indeed, it is now well-known, in the case of Stedman, to be false, i.e. there are bobs-only extents of Stedman Triples [11, 17] and [1, p. 276]. Burbidge [1, p. 263] says that

> "What must have convinced those who set themselves against the possibility of a bobs only peal must surely have been the age old problem in composition, . . . , that of being able to add only an even number of blocks to a piece using bobs, and therefore starting with one block you must always end up with an odd number of blocks, whereas the peal you seek is one of an even number of blocks. This is in essence why [an extent of] Grandsire Triples is not attainable by bobs only."

Note that in fact an extent (if it existed) would be of an odd number (i.e. 1) of blocks, and the starting position is of an even number, namely 72, of plained blocks.

---

[22] Note that $0 - 1 = 4 \pmod 5$. We might also define β as $(\gamma')^{-1}$.

[23] For example, suppose that before the bobbing we have the two round blocks $C_0C_1C_3$ and C_2C_4. After the bobbing, we have the two round blocks $C_0C_2C_3C_4$ and C_1. So α is (in full cycle notation) (013)(24) and β is (0234)(1). In this case we haven't changed the number of round blocks. As another example, suppose that before the bobbing we have the five round blocks C_0, C_1, C_2, C_3 and C_4. After the bobbing we have just the round block $C_1C_0C_4C_3C_2$. So α is (in full cycle notation) (0)(1)(2)(3)(4) and β is (04321); we have reduced the number of round blocks by 4.

[24] The touch PPPBPPPBPPPB seems to be what he has in mind; after 4 leads the row is 1342567, and after another 4 it is 1423567. But the length of the touch is 2.4 times the length of a plain course, rather than 3 times the length.

10 Long Bobs-only Touches of Grandsire Triples

Thompson [13] also conjectures, and argues for the conjecture, that there is no bobs-only touch of Grandsire Triples of length greater than 4998 (i.e. 5040 less 3×14) changes. His argument is that such a touch (of length 5012 or 5026 changes) must make a round block (of length 358 or 359 rows), and that the remaining rows must also make a round block; but, the shortest round block is a Bob course, of length 3, i.e. too long to fit.

We are not convinced by this argument. The problem is not that the touch must make a round block (that's obvious) but that the remainder must make one; this isn't obvious. (It is the case for Holt's touch of 4998 changes, i.e. 357 rows, the remainder being just a Bob course. But one example doesn't prove a general rule.)

The essence of Thompson's argument about an **extent** is that if one row x is followed by a Bob, (i.e. is 'Bobbed') then **all** the other four rows Q-equivalent to x must, to avoid falsity of the extent, be Bobbed; this then has predictable effects on the number of round blocks. This argument, so far as it concerns extents, is correct.

But, now consider a row x in a **touch** and the four other rows xQ, xQ^2, xQ^3 and xQ^4 in the same Q-set. Can some of these rows be Plained and others be Bobbed, without making the touch false? Well, if some of them are Absent from the touch, yes. The following two simple conditions must hold:

1. If in a touch x occurs Plained, then xP occurs next in the touch; so, if $xQ = xPB^{-1}$ is in the touch it must be Plained.

2. If in a touch x occurs Bobbed, then xB occurs next in the touch; so, if $xQ^{-1} = xBP^{-1}$ is in the touch it must be Bobbed.

The second of these is in fact an immediate consequence of the first; they are equivalent. We'll re-present the first as "if, in a touch, the row x occurs Plained, then xQ is either Plained or Absent from the touch". This proves the following result:

Theorem 10.1. *Let T be a touch; from each Q-cycle is determined thereby a cyclic sequence (of length 5) of labels P, B and A, the last being for* Absent. *The touch being true, i.e. repetition-free, no such cyclic sequence can contain the label P immediately followed by B.*

But we can have (for example) y Plained, yQ Absent, yQ^2 Absent, yQ^3 Bobbed and yQ^4 Absent. A real example is required to make matters clear.

Example 10.1. *Consider the touch $PPPPBPPBPBBPPPBPPBPB$ of* 20 *rows (i.e. with 280 changes).*

Let us number the 21 rows (with rounds added at the start) as $r_0, r_1, r_2, \ldots, r_{20}$. Row 10 is $r_{10} = 1672534$, which is Q-equivalent to $r_0 = 1234567$, with $r_{10} = r_0Q^3$. Row r_0 is Plained, but row r_{10} is Bobbed. The relevant Q-cycle is $(1234567, 1346572, 1467523, 1672534, 1723546)$*; according to this touch, it is annotated with* (P, A, A, B, A). *In other words,* 1234567 *is Plained,* 1346572 *is Absent,* 1467523 *is Absent,* 1672534 *is Bobbed and* 1723546 *is Absent. The combination P, B does not occur in this Q-cycle's annotation.*

In the following, the first column consists of the row numbers, the next column consists of the actual rows, the next column consists of the canonical Q-representatives of the rows, and the next number is the number of Qs used to get from the representative to the actual row. Then, optionally, there is a B. Then the same applies to the remaining columns. We write the symbol 'B' (for Bobbed) on the **right** *rather than on the left; it means not "how the row is obtained" but "what is done to it". Other rows are, by default, 'Plained' (unless they are 'Absent'). 17 different Q-sets are involved: note that $r_4 \equiv_Q r_{19}$ and $r_8 \equiv_Q r_{11}$. The full touch is as follows:*

```
 0  [1,2,3,4,5,6,7]    [1,2,3,4,5,6,7]  0
 1  [1,2,5,3,7,4,6]    [1,3,4,6,7,2,5]  3             11  [1,4,5,6,3,7,2]    [1,6,7,2,3,4,5]  3
 2  [1,2,7,5,6,3,4]    [1,2,7,5,6,3,4]  0             12  [1,4,3,5,2,6,7]    [1,5,6,7,2,4,3]  3
 3  [1,2,6,7,4,5,3]    [1,7,5,3,4,2,6]  3             13  [1,4,2,3,7,5,6]    [1,3,5,6,7,4,2]  3
 4  [1,2,4,6,3,7,5]    [1,7,5,2,3,4,6]  2 B           14  [1,4,7,2,6,3,5]    [1,2,3,5,6,4,7]  3 B
 5  [1,5,3,2,7,4,6]    [1,2,4,6,7,5,3]  3             15  [1,5,6,4,3,7,2]    [1,4,7,2,3,5,6]  3
 6  [1,5,7,3,6,2,4]    [1,2,4,5,6,7,3]  2             16  [1,5,3,6,2,4,7]    [1,6,4,7,2,5,3]  3
 7  [1,5,6,7,4,3,2]    [1,6,7,3,4,2,5]  4 B           17  [1,5,2,3,7,6,4]    [1,2,3,6,7,4,5]  4 B
 8  [1,2,4,5,3,6,7]    [1,6,7,2,3,4,5]  2             18  [1,4,7,5,6,2,3]    [1,4,7,5,6,2,3]  0
 9  [1,2,3,4,7,5,6]    [1,4,5,6,7,2,3]  3 B           19  [1,4,6,7,3,5,2]    [1,7,5,2,3,4,6]  3 B
10  [1,6,7,2,5,3,4]    [1,2,3,4,5,6,7]  3 B           20  [1,2,3,4,5,6,7]    [1,2,3,4,5,6,7]  0
```

We see this as a **local** *counterexample*[25] *to the argument in [13] and elsewhere for the conjecture that there can be no touch of length greater than 357 rows (i.e. 4998 changes); that argument, if correct, would show that,*

[25] Thus it refutes the argument, i.e. shows it to be incorrect; it doesn't refute the conjecture.

in a touch, in each Q-set, all rows are either all Plained or all Bobbed, and this is now seen to be false. So, Thompson's argument is incorrect.

However, in June 2016, van der Sluijs [12] provided a proof of the conjecture, using the new concept of an extended Q-set. The present paper's contribution, therefore, is not to express doubt about the truth of the conjecture, but to show that Thompson's argument for it in [13] is unsound. [12] is a novel contribution to the literature—but it hardly explains what exactly is wrong with the argument in [13].

11 Coherence

We define a collection $\mathcal{C}$ of round blocks to be *coherent* iff (i) the blocks are disjoint and (ii) whenever its union intersects a Q-set, the elements of the intersection are either all Plained or all Bobbed. Otherwise it is *incoherent*. Clearly a decomposition is coherent.

Theorem 4. If a collection $\mathcal{C}$ of round blocks is coherent, then it can be extended to a decomposition $\mathcal{D}$.

Proof. $\mathcal{C}$ determines for some Q-sets that they are Plained and for some that they are Bobbed; any other Q-sets can be fixed as (for example) Plained. The Plaining or Bobbing of each row is now determined, so by Theorem 1 we have a decomposition $\mathcal{D}$, which is easily seen to extend $\mathcal{C}$. **QED**.

This applies in particular to a collection consisting of a single block, and even more particularly to a touch. It is clear that an incoherent touch (such as that, of length 20 rows, analysed in Section 6) cannot be extended to a decomposition.

Experiments show that, below length 20, all the touches (227 in all) are (surprisingly) coherent. There are 757 touches of length at most 20, of which 557 are coherent and 200 (i.e. 26%) are incoherent. For length at most 25, there are 10,924 touches, of which 7,617 are coherent and 3,307 (i.e. 30%) are incoherent. We have no experimental results beyond length 25, except that, as one also infers from the above theory, Holt's touch [2, Touch 89] of 357 rows (the last 3 rows of his 360 rows being omitted by our avoidance of Singles) is coherent: the remaining block needed to make a decomposition is just a Bob course with a suitable starting point..

12 Extended Q-sets

This section is heavily based on [12], but adapted to ensure conformity with our own notation and conventions about multiplication of operators.

Let O be an operator: then $\mathbf{O}$ means the set of operators, usually written $\langle O\rangle$, generated by O. For example, if O has order 5, then it is $\{I, O, O^2, O^3, O^4\}$. It is in fact a group, but we don't need this fact. If x is a row, then $x\mathbf{O}$ is just the set $\{x, xO, xO^2, xO^3, xO^4\}$. Some of the arithmetic in what follows is done mod 5. Recall that $Q = PB^{-1}$, that $QB = P$ and that $Q^4P = B$.

Let $S = B^{-1}P = B^{-1}QB = P^{-1}QP = (25764)$; so $BS = P$ and the following identities are easy: $Q^{i+1}B = Q^iP = BS^{i+1} = PS^i$. It follows that $\mathbf{Q}B = \mathbf{Q}P = B\mathbf{S} = P\mathbf{S}$.

The *extended Q-set* E_x of a row x is the union of the Q-set of x and what we might call the *S-set* of xP, i.e. is

$$x\mathbf{Q} \cup x\mathbf{Q}B = x\mathbf{Q} \cup x\mathbf{Q}P = x\mathbf{Q} \cup xB\mathbf{S} = x\mathbf{Q} \cup xP\mathbf{S}.$$

Since B is not in $\mathbf{Q}$, each extended Q-set E_x is the union of two disjoint sets each of order 5, the first of which is a Q-set and the second of which is not a Q-set but an S-set and has several convenient descriptions.

Remark 12.1. *[12]. Let x be a row and R a round block. Then R contains equal numbers of elements from $x\mathbf{Q}$ and $x\mathbf{Q}B = x\mathbf{Q}P = xB\mathbf{S} = xP\mathbf{S}$.*

Proof. [26] Let R be a round block and let the row $y = xQ^i$ be in $x\mathbf{Q}$. If y is plained in R, then $xQ^iP = xPS^i$ is in R. If y is bobbed in R, then $xQ^iB = xPS^{i-1}$ is in R. Likewise, if the row $z = xPS^i$ is in $xP\mathbf{S}$ and in R, then it is in R either as wP or as wB for some unique w in R. In the first case $w = xPS^iP^{-1} = xQ^i$ is in $x\mathbf{Q}$; in the second case $w = xPS^iB^{-1} = xQ^{i+1}$ is again in $x\mathbf{Q}$. This gives us a 1-1 correspondence between the intersections of the sets $x\mathbf{Q}$ and $xP\mathbf{S}$ with R; thus they have the same number of members in R. □

Remark 12.2. *[12]. Each row x is contained in exactly two extended Q-sets: E_x and $E_{xB^{-1}}$.*

[26] The proof in [12] is based on the use of a Cayley graph.

Proof. x is certainly in E_x; that it is also in $E_{xB^{-1}}$ follows from the identities $xB^{-1}P\mathbf{S} = xB^{-1}B\mathbf{S} = x\mathbf{S}$. That these are different follows from the occurrence of xP in the first but not the second: if xP is in $xB^{-1}\mathbf{Q}$, then $BP = (2637)(45)$ is a power of Q (which is not the case), and similarly if it were in $xB^{-1}B\mathbf{S}$, we could infer that P is a power of S, likewise not the case.

Suppose also that x is in E_y. There are two cases:

1. $x \in y\mathbf{Q}$, hence $y\mathbf{Q} = x\mathbf{Q}$, hence $E_y = y\mathbf{Q} \cup y\mathbf{Q}B = x\mathbf{Q} \cup x\mathbf{Q}B = E_x$;
2. $x \in y\mathbf{Q}B$, hence $y\mathbf{Q} = xB^{-1}\mathbf{Q}$, hence $E_y = y\mathbf{Q} \cup y\mathbf{Q}B = xB^{-1}\mathbf{Q} \cup xB^{-1}\mathbf{Q}B = E_{xB^{-1}}$. □

Theorem 12.1. *[12]. There is no round block of length greater than 357.*

Proof. [12]. Let R be a round block. By the Corollary to Theorem 2 there is a row x not in R. Let $x' = xB^{-1}$ and let $E_x = x\mathbf{Q} \cup x\mathbf{Q}B$ and $E_{x'} = xB^{-1}\mathbf{Q} \cup xB^{-1}\mathbf{Q}B$ be the two extended Q-sets to which x belongs. We will show that $E_x \cap E_{x'} = \{x\}$. This follows from the following equations:

(1) $\quad \mathbf{Q} \cap B^{-1}\mathbf{Q} = \emptyset$

(2) $\quad \mathbf{Q} \cap B^{-1}\mathbf{Q}B = \mathbf{Q} \cap \mathbf{S} = (\mathbf{27643}) \cap (\mathbf{25764}) = \{I\}$

(3)
$$\begin{aligned}\mathbf{Q}B \cap B^{-1}\mathbf{Q} &= \mathbf{Q}B \cap (\mathbf{Q}B)^{-1} \\ &= (\mathbf{27643})(247)(365) \cap ((\mathbf{27643})(247)(365))^{-1} \\ &= \{B, P, (253), (2764)(35), (26)(3745)\} \cap \{B, P, (253), (2764)(35), (26)(3745)\}^{-1} \\ &= \emptyset\end{aligned}$$

(4) $\quad \mathbf{Q}B \cap B^{-1}\mathbf{Q}B = \emptyset$

where (1) holds, as already mentioned above, because $B \notin \mathbf{Q}$; (2) holds because the two cycles shown fix positions 5 and 3 respectively; (3) holds by computation and consideration of the cycle structures; and (4) holds as a consequence of (1).

Now, by Remark 1, $x\mathbf{Q} \cap R$ and $x\mathbf{Q}B \cap R$ are the same size; and so, since $x\mathbf{Q}$ and $x\mathbf{Q}B$ are the same size, $x\mathbf{Q}\backslash R$ and $x\mathbf{Q}B\backslash R$ are the same size. x is in the first, so we can find $y \neq x$ in the second. Similarly we can find $z \neq x$ in $xB^{-1}\mathbf{Q}\backslash R$. By Remark 2, $y \neq z$. So we have three rows x, y and z not in R. □

The following seems to be new (in the sense that, although Thompson stated the result, his proof is incorrect), but matches, we believe, the experience of those such as Holt who composed touches of length 357. If the composition is done by maintaining the coherence of the touch, then it is immediate; but, as seen above, touches need not be coherent.

Theorem 12.2. *Let R be a touch of length 357. Then the remaining three rows also form a round block.*

Proof. By Thompson's Theorem and the argument just given, we can find distinct rows x, y and z not in R. The present theorem's hypothesis is that there are no more. By construction, there exist i and j with both $y = xBS^i$ and $z = xB^{-1}Q^j$ not in R. Applying the same argument again to y, we can find k with $w = xBS^iBS^k$ distinct from y and not in R. But there are only three possibilities; so w must equal one of x and z. Suppose that $w = x$; then $BS^iBS^k = I$; so $S^iQ^kB = S^iBS^k = B^2$, whence $B = S^iQ^k$ and $Q^k = S^{5-i}B$. S^{5-i} fixes the bell in position 3; B moves it to position 6; if Q^k does the same then $k = 3$. But then BQ^k must be $(247)(365)(24736) = (2743)(56)$, not a power of S. So we must instead have $w = z$, i.e. $xBS^iBS^k = xB^{-1}Q^j$, whence

$$B^{-1}S^iBS^k = B^2S^iBS^k = Q^j.$$

This is satisfied when $i = j = k = 0$; we show that there are no other possibilities. Recall that $S = (25764)$ and $Q = (27643)$; so $S^{-1}Q = (23)(57)$, $SQ^{-1} = (25)(34)$ and $S^3Q^3 = (36574)$.

Now, $B^{-1}S^iB = (274)(356)(25764)^i(247)(365)$ fixes the bell in position 6. S^k, for $k = 0, 1, 2, 3$ and 4, takes the bell in position 6 to positions 6, 4, 2, 5 and 7, respectively. Meanwhile, Q^j (for $j = 0, 1, 2, 3, 4$) takes the bell in position 6 to positions 6, 4, 3, 2 and 7 respectively. So, $B^{-1}S^iBS^k$ and Q^j can only be equal if, excluding $j = k = 0$, we have $k = j = 1$ or $k = j - 1 = 2$ or $k = j = 4$. The first case is impossible, since $B^{-1}S^iB$ is (for $i \neq 1$), a 5-cycle and $S^{-1}Q = (23)(57)$; the second case is impossible, since $S^{-2}Q^3 = (36574)$, which is not a power of $B^{-1}SB = (25743)$; the third case is impossible, since $S^{-4}Q^4 = SQ^{-1} = (25)(34)$ is also not a 5-cycle. There remains the possibility, now shown to be a certainty, of $i = j = k = 0$; and so the three rows form the round block (x, xB, xB^2). □

Of course, now we know that the remainder is a round block, the two round blocks must form a decomposition and the touch must be coherent.

13 Possible further work

As we have seen, the methods used for Grandsire Triples show that there is no bobs-only extent of Plain Bob Major. We conjecture (without any evidence) (i) that the longest bobs-only touch is of length at most $40,240$ changes (i.e. 2,515 leads); (ii) that there is such a composition of exactly 2,515 leads; and (iii) that the remaining five leads form a bob course (of length 80 changes). Whether anyone has the patience and skill to demonstrate (ii) remains to be seen; but, if it can be, then, since the rightly celebrated achievement of 40,320 (composed by C Kenneth Lewis) at Loughborough in 1963, it is not as impossible as in 1887 that it might, one day, be rung.

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